Properties of Limits

-The following is a list of properties that can be applied to limits. NOTE: For any real number c, suppose the functions f and g both have limits at $x = c$

Constant Rule

$$\lim_{x \to c} k = k \text{ for any constant } k$$

Limit of x Rule

$$\lim_{x \to c} \left[s \cdot f(x) \right] = s \cdot \left[\lim_{x \to c} f(x) \right]$$

"The limit of a constant times a function is the constant times the limit of a function."

Example: $\lim_{x \to 0} 2 \cdot \sin(x) = 2 \cdot \lim_{x \to 0} \sin(x) = 2 \cdot 0 = 0$

Sum/Difference Rule

$$\lim_{x \to c} \left[f(x) \pm g(x) \right] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

"The limit of a sum is the sum of the limits"

Example: $\lim_{x \to c} f(x) = 7 \qquad \lim_{x \to c} g(x) = 4$

$$\lim_{x \to c} \left[f(x) + g(x) \right] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = 7 + 4 = 11$$

Product Rule

$$\lim_{x \to c} \left[f(x) \cdot g(x) \right] = \left[\lim_{x \to c} f(x) \right] \left[\lim_{x \to c} g(x) \right]$$

"The limit of a difference is the difference of the limits"

Quotient Rule

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \quad \text{if} \quad \lim_{x \to c} g(x) \neq 0$$

"The limit of a quotient is the quotient of the limits, as long as the limit of the denominator is not zero."

Power Rule

$$\lim_{x \to c} \left[f(x) \right]^n = \left[\lim_{x \to c} f(x) \right]^n \qquad \text{Note: } n \text{ is a rational number}$$

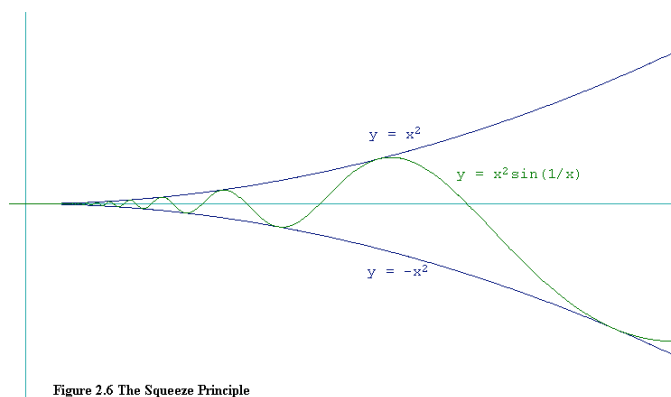"The limit of a power is the power of the limit."

Squeeze Rule

If on some interval around c

$$g(x) \leq f(x) \leq h(x) \qquad \text{and} \qquad \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

Then

$$\lim_{x \to c} f(x) = L$$

"If a function can be squeezed between two functions with equal limits, then that function must also have the same limit."

$y = x^2$

$y = x^2 \sin(1/x)$

$y = -x^2$

Figure 2.6 The Squeeze Principle

Limit of Polynomial Functions

Evaluate $\lim_{x \to 2} \left(2x^5 - 9x^3 + 3x^2 - 11 \right)$

$$= \lim_{x \to 2} 2x^5 - \lim_{x \to 2} 9x^3 + \lim_{x \to 2} 3x^2 - \lim_{x \to 2} 11$$

$$= 2\lim_{x \to 2} x^5 - 9\lim_{x \to 2} x^3 + 3\lim_{x \to 2} x^2 - 11$$

$$= 2\left[\lim_{x \to 2} x \right]^5 - 9\left[\lim_{x \to 2} x \right]^3 + 3\left[\lim_{x \to 2} x \right]^2 - 11$$

$$= 2(2)^5 - 9(2)^3 + 3(2)^2 - 11$$
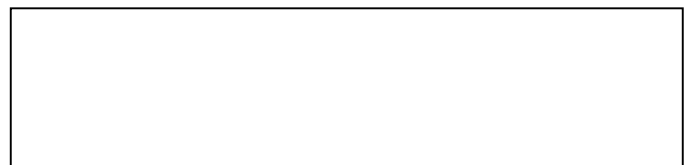
$$= -7$$

-How else could this have been accomplished?

-If f is a polynomial the $\lim_{x \to c} f(x)$ can be found wherever $x = c$. Substitute c in for x and solve.

Limits of Rational Functions

Evaluate $\lim_{z \to -1} \dfrac{z^3 - 3z + 7}{5z^2 + 9z + 6}$

$$= \frac{\lim_{z \to -1} \left(z^3 - 3z + 7 \right)}{\lim_{z \to -1} \left(5z^2 + 9z + 6 \right)}$$

$$= \frac{(-1)^3 - 3(-1) + 7}{5(-1)^2 + 9(-1) + 6} = \frac{9}{2}$$

Limits of a power (or root) function

Evaluate $\displaystyle\lim_{x\to-2} \sqrt[3]{x^2 - 3x - 2}$

$$= \lim_{x\to-2} \left(x^2 - 3x - 2\right)^{1/3}$$

$$= \left[\lim_{x\to-2} \left(x^2 - 3x - 2\right)\right]^{1/3}$$

$$= \left[\left(-2\right)^2 - 3\left(-2\right) - 2\right]^{1/3} = 8^{1/3} = 2$$

Finding Trigonometric Limits Algebraically

Given $\displaystyle\lim_{x\to0} \sin x = 0$ and $\displaystyle\lim_{x\to0} \cos x = 1$

a) $\displaystyle\lim_{x\to0} \sin^2 x$

$$= \left[\lim_{x\to0} \sin x\right]^2 = 0^2 = 0$$

b) $\displaystyle\lim_{x\to0} \left(1 - \cos x\right)$

$$= \lim_{x\to0} 1 - \lim_{x\to0} \cos x$$

$$= 1 - 1 = 0$$

Using Algebra to Find Limits

-Sometimes substitution will not solve the limit and we can use algebra to help.

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \frac{(2)^2 + 2 - 6}{2 - 2} = \frac{0}{0} \otimes$$

-A case such as this is known as an <u>indeterminate form</u>.
-We need only examine x as it approaches 2 NOT actually at 2 so this works.

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x + 3)(x - 2)}{x - 2}$$

$$= \lim_{x \to 2} (x + 3) = 5$$

Evaluating Limits by Rationalizing

Evaluate $\displaystyle \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{4} - 2}{4 - 4} = \frac{0}{0} \otimes$

$$= \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \bullet \frac{\sqrt{x} + 2}{\sqrt{x} + 2} \leftarrow \text{conjugate}$$

$$= \lim_{x \to 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)}$$

$$= \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

Evaluating Trig Functions with Conjugate

Evaluate $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x}$

$$= \lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} \bullet \dfrac{1 + \cos x}{1 + \cos x}$$

$$= \lim\limits_{x \to 0} \dfrac{1 - \cos^2 x}{x\left(1 + \cos x\right)}$$

$$= \lim\limits_{x \to 0} \dfrac{\sin^2 x}{x\left(1 + \cos x\right)}$$

$$= \lim\limits_{x \to 0} \left[\left(\dfrac{\sin x}{x}\right)\left(\dfrac{\sin x}{1 + \cos x}\right)\right]$$

$$= \lim\limits_{x \to 0} \dfrac{\sin x}{x} \bullet \lim\limits_{x \to 0} \dfrac{\sin x}{1 + \cos x}$$

$$= 1 \bullet 0$$

$$= 0$$

Limits of Piecewise-Defined Functions

Evaluate $\lim\limits_{x \to 0} f\left(x\right) = \begin{cases} x + 5 & \text{if } x > 0 \\ x & \text{if } x < 0 \end{cases}$

-Because $f\left(0\right)$ is not defined we will have to evaluate the limit from both sides.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x + 5 = 0 + 5 = 5$$

-Because the left and right handed limits are different we say $\lim_{x \to 0} f(x)$
does not exist.

Example

Evaluate $\lim_{x \to 0} g(x)$ where $g(x) = \begin{cases} x + 1 & \text{if } x > 0 \\ x^2 + 1 & \text{if } x < 0 \end{cases}$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} x^2 + 1 = 0^2 + 1 = 1$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} x + 1 = 0 + 1 = 1$$

-Since the limit from the left and the right are the same we say
$\lim_{x \to 0} g(x) = 1$.